INDESTRUCTIBILITY OF COMPACT SPACES

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Abstract. In this article we investigate which compact spaces remain compact under countably closed forcing. We prove that, assuming the Continuum Hypothesis, the natural generalizations to ω_1-sequences of the selection principle and topological game versions of the Rothberger property are not equivalent, even for compact spaces. We also show that Tall and Usuba's "$\aleph_1$-Borel Conjecture" is equiconsistent with the existence of an inaccessible cardinal.

1. Introduction

The question of whether the Lindelöf property of topological spaces is preserved by countably closed forcing is of interest in conjunction with A. V. Arhangel'skiĭ's classic problem of whether Lindelöf T_2 spaces with points G_δ have cardinality not exceeding the continuum [1]. For a survey of this problem, see [33].

Definition 1.1 (Tall [33])**.** *A Lindelöf space is* indestructible *if the topology it generates in any countably closed forcing extension is Lindelöf.*

Theorem 1.2 (Tall [33])**.** *Lévy-collapse a supercompact cardinal to ω_2 with countable conditions. Then every indestructible Lindelöf space with points G_δ has cardinality $\leq \aleph_1$.*

This was later improved by M. Scheepers [27], who replaced "supercompact" by "measurable". "Points G_δ" was then improved to "pseudocharacter $\leq \aleph_1$" by Tall and T. Usuba in [35].

Scheepers and Tall [28] noticed that indestructibility of a Lindelöf space is equivalent to player One not having a winning strategy in an ω_1-length generalization of the *Rothberger game* introduced by F. Galvin in [9].

Definition 1.3. *Let X be a topological space and α be an ordinal. We denote by $\mathsf{G}_1^\alpha(\mathcal{O}_X, \mathcal{O}_X)$ the game defined as follows. In each inning $\xi \in \alpha$, player One chooses an open cover $\mathcal{U}_\xi$ of X, and then player Two picks $V_\xi \in \mathcal{U}_\xi$. Two wins the play if $X = \bigcup\{V_\xi : \xi \in \alpha\}$; otherwise, One is the winner.*

Theorem 1.4 (Scheepers-Tall [28])**.** *A Lindelöf space X is indestructible if and only if One does not have a winning strategy in the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$.*

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Definition 1.5. *Let X be a topological space and α be an ordinal.* $\mathsf{S}_1^\alpha(\mathcal{O}_X, \mathcal{O}_X)$ *denotes the following statement: "For every sequence $(\mathcal{U}_\xi)_{\xi\in\alpha}$ of open covers of X, there is a sequence $(U_\xi)_{\xi\in\alpha}$ such that $U_\xi \in \mathcal{U}_\xi$ for each $\xi \in \alpha$ and $X = \bigcup\{U_\xi : \xi \in \alpha\}$".*

Note that $\mathsf{S}_1^\omega(\mathcal{O}_X, \mathcal{O}_X)$ means that X is a Rothberger space [25].

It is easily seen that the nonexistence of a winning strategy for player One in $\mathsf{G}_1^\alpha(\mathcal{O}_X, \mathcal{O}_X)$ implies $\mathsf{S}_1^\alpha(\mathcal{O}_X, \mathcal{O}_X)$. In [23], J. Pawlikowski proved the converse of this implication for the case $\alpha = \omega$:

Theorem 1.6 (Pawlikowski [23])**.** *A topological space X is Rothberger if and only if One has no winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}_X, \mathcal{O}_X)$.*

We thank Boaz Tsaban for the following observation. Pawlikowski's result yields the analogous equivalence for $\omega \le \alpha < \omega_1$:

Corollary 1.7. *Let α be an infinite countable ordinal. The following are equivalent for a topological space X:*

- (a) *One does not have a winning strategy in $\mathsf{G}_1^\alpha(\mathcal{O}_X, \mathcal{O}_X)$;*
- (b) $\mathsf{S}_1^\alpha(\mathcal{O}_X, \mathcal{O}_X)$*;*
- (c) *X is Rothberger.*

Proof. We have already pointed out that $(a) \to (b)$. The equivalence between (b) and (c) is immediate since $|\alpha| = |\omega|$. Finally, if every strategy for One in the game $\mathsf{G}_1^\omega(\mathcal{O}_X, \mathcal{O}_X)$ can be defeated, then clearly the same holds in the longer game $\mathsf{G}_1^\alpha(\mathcal{O}_X, \mathcal{O}_X)$; therefore, (c) implies (a) in view of Theorem 1.6. □

In light of Theorems 1.4 and 1.6, Scheepers and Tall asked whether the corresponding equivalence would also hold for the case $\alpha = \omega_1$ [28, 34]:

Problem 1.8 (Scheepers-Tall [28], Tall [34])**.** *Is indestructibility of a Lindelöf space X equivalent to $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$?*

We partially answer their question by showing (in Section 3):

Theorem 1.9. *There is a compact T_2 destructible space that, assuming the Continuum Hypothesis, satisfies $\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$.*

The fact that this example is compact leads us to investigate the indestructibility of compact spaces. Notice that:

Lemma 1.10. *A compact space is indestructible if and only if it remains compact under countably closed forcing.*

Proof. This follows from the observation that countable compactness is preserved by countably closed forcing. □

We shall also prove (in Corollaries 4.5 and 3.3, respectively) that:

Theorem 1.11. *Dyadic spaces of cardinality greater than $\mathfrak{c}$ are destructible.*

Theorem 1.12. *Compact T_2 spaces in which no point is a G_δ are destructible.*

On the other hand, I. Juhász and W. Weiss proved in [15] that:

Theorem 1.13 (Juhász-Weiss [15])**.** *Lindelöf scattered spaces are indestructible.*

In particular,

Example 1.14. *The one-point compactification of any discrete space is indestructible.*

Here it is worth mentioning that a compact Hausdorff space is scattered if and only if it is Rothberger (folklore; see *e.g.* [2, Proposition 5.5]).

We also have:

Theorem 1.15. *The Continuum Hypothesis implies that if a compact T_2 space has all points G_δ, then it is indestructible.*

Proof. We first recall:

Lemma 1.16 (Tall [33]). *Lindelöf spaces of size $\leq \aleph_1$ are indestructible.*

Proof. If $X = \{x_\xi : \xi \in \omega_1\}$, then Two gets a winning strategy in $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ by covering the point x_ξ in the ξ-th inning. Now apply Theorem 1.4. □

Now note that compact Hausdorff spaces with all points G_δ are first countable and so have cardinality $\leq \mathfrak{c}$ by Arhangel'skiĭ's Theorem [1]; thus, if CH holds, such spaces are indestructible by Lemma 1.16. □

When some (but not all) points are G_δ's the situation is less clear. A new direction for dealing with this problem was provided by Tall and Usuba [35] through a "one cardinal up" version of Borel's Conjecture [5, p. 123].

By results of W. Sierpiński [31] and R. Laver [21], Borel's Conjecture is independent of ZFC. It is also known (see [22, Proposition 8] and [4, Section 5.1]) to be equivalent to the statement that a Lindelöf T_3 space is Rothberger if and only if all of its continuous images in $[0,1]^\omega$ are countable. In light of this result and Theorems 1.6 and 1.4, Tall and Usuba formulated the following statement in analogy to Borel's Conjecture:

Definition 1.17 (Tall-Usuba [35]). *The* $\aleph_1$-Borel Conjecture *is the statement that a Lindelöf T_3 space is indestructible if and only if all of its continuous images in $[0,1]^{\omega_1}$ have cardinality $\leq \aleph_1$.*

Techniques from S. Todorčević [37] concerning trees and their associated lines enable us to answer questions of [35]; in particular, we show (in Corollary 5.11):

Theorem 1.18. *The $\aleph_1$-Borel Conjecture and the existence of an inaccessible cardinal are equiconsistent.*

2. Preliminaries

Our notation and terminology are standard; that said, in this section we include some definitions and summarize some known results for the convenience of the reader. For concepts not defined in this section, the reader is referred to [18], [8] and [16].

A forcing notion $\langle \mathbb{P}, \leq \rangle$ is *countably closed* if, whenever α is a countable ordinal and $(p_\xi)_{\xi \in \alpha}$ is a decreasing sequence in $\mathbb{P}$, there is $q \in \mathbb{P}$ such that $q \leq p_\xi$ for all $\xi \in \alpha$. In our forcing arguments, $\mathbf{M}$ will always denote the ground model. If $\langle X, \mathcal{T} \rangle$ is a topological space in $\mathbf{M}$ and $\mathbb{P}$ is a forcing notion, in a generic extension $\mathbf{M}[G]$ by $\mathbb{P}$ the set $\mathcal{T}$ might fail to be a topology, so we always consider the corresponding space $\langle X, \widetilde{\mathcal{T}} \rangle$, where $\widetilde{\mathcal{T}}$ is the topology on X (in the extension $\mathbf{M}[G]$) that has $\mathcal{T}$ as an open base.

If α is an ordinal, we write $\lim(\alpha) = \{\xi \in \alpha : \xi \text{ is a limit ordinal}\}$. We denote by $\mathfrak{c}$ the cardinality of the continuum, *i.e.* $2^{\aleph_0}$; the Continuum Hypothesis (CH) is the assertion $\mathfrak{c} = \aleph_1$.

A *Dedekind cut* in a linear order $\langle X, <\rangle$ is a set $A \subseteq X$ such that $y \in A$ whenever $y < x$ for some $x \in A$. The linear order $\langle X, <\rangle$ is *complete* if every Dedekind cut has a least upper bound.

For a set X and a cardinal κ, we write $[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\}$ and $[X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}$. A family $\mathcal{F}$ of sets is *centred* if $\bigcap \mathcal{A} \neq \emptyset$ for every $\mathcal{A} \in [\mathcal{F}]^{<\aleph_0} \setminus \{\emptyset\}$. The set of all functions from a set A to a set B is denoted by ${}^A B$; if α is an ordinal, then we write ${}^{<\alpha}B = \bigcup_{\xi\in\alpha} {}^\xi B$ and ${}^{\leq\alpha}B = {}^{<\alpha+1}B$. If A and B are nonempty sets, we write $Fn(A, B) = \bigcup\{{}^F B : F \in [A]^{<\aleph_0}\}$. The two-point set $\{0, 1\}$ will be denoted simply by 2, and will always be regarded as a discrete space. If $X = \prod_{i\in I} X_i$ is a product, then for each $j \in I$ the function $\pi_j : X \to X_j$ is the projection $\pi_j((x_i)_{i\in I}) = x_j$.

If X is a topological space and $p \in X$, we say that p is a G_δ point of X if the set $\{p\}$ is a countable intersection of open subsets of X. The *weight* of a space X is $w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\} + \aleph_0$. The *pseudocharacter* of a T_1 space X is $\psi(X) = \sup\{\psi(x, X) : x \in X\}$, where $\psi(x, X) = \min\{\kappa : \{x\} \text{ is an intersection of } \kappa$ open subsets of $X\} + \aleph_0$; for compact T_2 spaces, $\psi(x, X) = \chi(x, X) = \min\{|\mathcal{V}| : \mathcal{V}$ is a local base at x in $X\} + \aleph_0$, the *character* of x in X. If $\langle X, \mathcal{T}\rangle$ is a topological space and $x \in X$, a π*-base* for x in X is a family $\mathcal{P} \subseteq \mathcal{T} \setminus \{\emptyset\}$ such that every neighbourhood of x in X includes some element of $\mathcal{P}$; the π*-character* of $\langle X, \mathcal{T}\rangle$, then, is $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$, where $\pi\chi(x, X) = \min\{|\mathcal{P}| : \mathcal{P}$ is a π-base for x in $X\} + \aleph_0 \leq \chi(x, X)$. A topological space X is said to include a copy of a topological space Y if it has a subspace homeomorphic to Y. We say that a topological space X is *scattered* if every nonempty $Y \subseteq X$ has a point that is isolated in Y. A topological space is *extremally disconnected* if the closure of every open subset is open.

A *tree* is a strict partial order $\langle T, \leq\rangle$ (often written simply T) such that, for each $t \in T$, the set $T^\downarrow(t) = \{t' \in T : t' < t\}$ is well-ordered. For each $t \in T$, the *height* of t in T, denoted by $\mathrm{ht}_T(t)$, is the order type of $T^\downarrow(t)$. If α is an ordinal, the α-th *level* of T is $T_\alpha = \{t \in T : \mathrm{ht}_T(t) = \alpha\}$; T is *rooted* if $|T_0| = 1$. The *height of* T is the least ordinal η with $T_\eta = \emptyset$. We say that T is *Hausdorff* if, whenever α is a limit ordinal and $t, t' \in T_\alpha$ are distinct, the sets $T^\downarrow(t)$ and $T^\downarrow(t')$ are distinct. A *subtree* of $\langle T, \leq\rangle$ is any subset $T' \subseteq T$ considered with the restriction of $\leq$ to $T' \times T'$ (which makes T' a tree as well); if a subtree T' is downwards closed, *i.e.* $T^\downarrow(t) \subseteq T'$ for all $t \in T'$, then T' is said to be an *initial part* of T.

A *chain* in T is any subset of T that is linearly ordered by $\leq$. A *branch* is a chain that is maximal with respect to inclusion of sets ($\subseteq$). The *cofinality* of a branch B is the least cardinality of a *cofinal* subset of B, *i.e.* a subset $C \subseteq B$ such that $\forall t \in B\ \exists t' \in C\ (t \leq t')$; equivalently, the cofinality of B is $\mathrm{cf}(\beta)$, where β is the order type of B with the order induced by $\leq$. A branch B is *cofinal* in T if $B \cap T_\alpha \neq \emptyset$ for all α with $T_\alpha \neq \emptyset$.

Suppose $\langle T, \leq\rangle$ is a Hausdorff tree where, for each $t \in T$, the set $S(t)$ of all the immediate successors of t in $\leq$ is linearly ordered by a relation $\prec_t$. The *lexicographical ordering* on the set of all the branches of T is the linear order $\prec$ defined by $B \prec B' \leftrightarrow v \prec_t v'$, where t is the $\leq$-greatest element of $B \cap B'$ and $v, v' \in S(t)$ are such that $v \in B$ and $v' \in B'$.

If κ is a cardinal, we say that T is a *κ-tree* if T has height κ and all the levels of T have size $< \kappa$. A *κ-Aronszajn tree* is a κ-tree with no cofinal branch. A *weak Kurepa tree* (or *Canadian tree*) is a tree of height ω_1, cardinality $\aleph_1$, and at least $\aleph_2$ cofinal branches. A *Kurepa tree* is an ω_1-tree with at least $\aleph_2$ cofinal branches. *Kurepa's Hypothesis* (KH) is the statement "there is a Kurepa tree".

An uncountable cardinal κ is *(strongly) inaccessible* if it is regular and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. If κ is inaccessible and there is no κ-Aronszajn tree, then κ is said to be *weakly compact*. It is known by a result of R. Solovay (see [37, Theorem 8.11]) that, if ω_2 is not inaccessible in the constructible universe $\mathbf{L}$, then KH holds. Whenever we say that two statements are equiconsistent, we mean that they are equiconsistent relative to ZFC.

3. Indestructibility *versus* $\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$

The following terminology is taken from [19]:

Definition 3.1. *Let κ be an infinite cardinal. A κ-*Čech-Pospíšil tree *in a topological space X is an indexed family $\langle F_s : s \in {}^{\leq\kappa}2\rangle$ satisfying:*

(i) each F_s is a nonempty closed subset of X;
(ii) $F_s \supseteq F_t$ whenever $s \subseteq t$;
(iii) $F_{s^\frown(0)} \cap F_{s^\frown(1)} = \emptyset$;
(iv) if $\gamma \leq \kappa$ is a nonzero limit ordinal and $s \in {}^\gamma 2$, then $F_s = \bigcap_{\alpha\in\gamma} F_{s\restriction\alpha}$.

Proposition 3.2. *If there is an ω_1-Čech-Pospíšil tree in a topological space X, then player One has a winning strategy in the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$.*

Proof. Let $\langle F_s : s \in {}^{\leq\omega_1}2\rangle$ be an ω_1-Čech-Pospíšil tree in X. For each $s \in {}^{\leq\omega_1}2$, consider $U_s = X \setminus F_s$. This defines the following strategy for player One in $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$: One starts the play with the open cover $\{U_{(0)}, U_{(1)}\}$; if $\alpha \in \omega_1$ and $s \in {}^\alpha 2$ are such that, for each $\beta \in \alpha$, Two's move in the β-th inning of this play was $U_{s\restriction(\beta+1)}$, then One's move in the α-th inning is $\{U_{s^\frown(0)}, U_{s^\frown(1)}\}$. When the play ends, Two will have played the sets $(U_{t\restriction(\beta+1)})_{\beta\in\omega_1}$ for some $t \in {}^{\omega_1}2$; thus One wins since $\bigcup\{U_{t\restriction(\beta+1)} : \beta \in \omega_1\} = U_t \neq X$. □

Corollary 3.3. *Every compact Hausdorff space with no G_δ points is destructible.*

Proof. It is a famous theorem [6] that, if X is a compact Hausdorff space with no G_δ points, then there is an ω_1-Čech-Pospíšil tree in X. The result then follows from Proposition 3.2 and Theorem 1.4. □

Corollary 3.4. *If an infinite compact Hausdorff space X does not contain nontrivial convergent sequences, then X is destructible.*

In the proof of Corollary 3.4, we shall make use of the following (probably folklore) fact:

Lemma 3.5. *If an infinite compact Hausdorff space X is scattered, then there is a nontrivial convergent sequence in X.*

Proof. Let $I \subseteq X$ be the set of isolated points of X. As X is compact Hausdorff, we cannot have $I = X$, for otherwise X would be finite. Thus $X \setminus I \neq \emptyset$. Since X is scattered, there are $x \in X \setminus I$ and a neighbourhood V of x such that $V \cap (X \setminus I) = \{x\}$, *i.e.* $V \subseteq I \cup \{x\}$. By regularity of X, we may assume that V is closed, hence

compact. Since $x \notin I$, we have that x is not isolated in V; in particular, V is infinite. It follows that V is homeomorphic to the one-point compactification of the infinite discrete set $V \cap I$; therefore, there is a nontrivial convergent sequence in V, and hence in X. $\square$

Proof of Corollary 3.4. Let $X = S\dot{\cup}P$ be the Cantor-Bendixson decomposition of X, where S is an open scattered subspace of X and P is a closed subspace of X no point of which is isolated (in P). If we had $P = \emptyset$, then X would be compact scattered T_2, and hence would have a nontrivial convergent sequence by Lemma 3.5; therefore, $P \neq \emptyset$. As no point of P is isolated, it follows that $\chi(x, P) \geq \aleph_1$ for all $x \in P$, since otherwise there would be a nontrivial sequence converging to a point of first countability of P. By Corollary 3.3, P is destructible; therefore, X is destructible since indestructibility is hereditary with respect to closed subspaces — see Lemma 4.2. $\square$

We now turn to the proof of Theorem 1.9.

For any distinct $f, g \in {}^{\omega_1}2$, define $\Delta(f, g) = \min\{\xi \in \omega_1 : f(\xi) \neq g(\xi)\}$. Let $\prec$ be the lexicographical ordering on the set ${}^{\omega_1}2$, *i.e.*, for any distinct $f, g \in {}^{\omega_1}2$ we have that $f \prec g$ if and only if $f(\Delta(f, g)) = 0$ and $g(\Delta(f, g)) = 1$. Finally, let X be the linearly ordered topological space ${}^{\omega_1}2$ obtained from the ordering $\prec$.

The following fact is well-known (see *e.g.* [10, Lemma 13.17]):

Lemma 3.6. *Every nonempty subset of X has a least upper bound and a greatest lower bound.*

As X is a linearly ordered space, Lemma 3.6 can be restated as:

Lemma 3.7. *X is compact.*

Next we will show that X satisfies the hypotheses of Corollary 3.3. In order to do so, we shall make use of the following two lemmas.

Lemma 3.8. *Let $f, g \in {}^{\omega_1}2$ be such that $f \prec g$. Then the open interval (f, g) is empty if and only if for every $\xi \in \omega_1$ with $\xi > \Delta(f, g)$ we have $f(\xi) = 1$ and $g(\xi) = 0$.*

Proof. See 13.16 in [10]. $\square$

Lemma 3.9. *Let $f \in {}^{\omega_1}2$.*

(*a*) *If $f = \sup A$, where $A \subseteq {}^{\omega_1}2$ is countable and has no greatest element, then the set $\{\xi \in \omega_1 : f(\xi) = 1\}$ is countable.*
(*b*) *If $f = \inf A$, where $A \subseteq {}^{\omega_1}2$ is countable and has no least element, then the set $\{\xi \in \omega_1 : f(\xi) = 0\}$ is countable.*

Proof. We prove (*a*); (*b*) is analogous.

Let $A \subseteq {}^{\omega_1}2$ be countable with no greatest element and such that $f = \sup A$. Since A has no greatest element, we have that $f \notin A$; let then $\delta = \sup\{\Delta(x, f) : x \in A\} + 1 \in \omega_1$. We claim that $f(\xi) = 0$ for all $\xi \in \omega_1 \setminus \delta$. Indeed, if $\alpha \in \omega_1 \setminus \delta$ were such that $f(\alpha) = 1$, we would have that the function $g \in {}^{\omega_1}2$ defined by

$$g(\xi) = \begin{cases} 0 & \text{if } \xi = \alpha \\ f(\xi) & \text{otherwise} \end{cases}$$

would be an upper bound for A satisfying $g \prec f$, thus contradicting the assumption that $f = \sup A$. $\square$

Lemma 3.10. *No point of X is a G_δ.*

Proof. Suppose that $f \in X$ is a G_δ point. Let us assume, for a moment, that f is nonconstant. Then there are sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X such that $\{g \in X : x_n \prec g \prec y_n \text{ for all } n \in \omega\} = \{f\}$. Let $x = \sup\{x_n : n \in \omega\}$ and $y = \inf\{y_n : n \in \omega\}$ — note that x and y are well-defined by virtue of Lemma 3.6. Since $x_n \prec f \prec y_n$ for all $n \in \omega$, we must have $x \preceq f \preceq y$.

Note that (†) $x \prec f$ implies $(x, f) = \emptyset$ and that (‡) $f \prec y$ implies $(f, y) = \emptyset$. Hence, in light of Lemma 3.8, $x \prec f \prec y$ cannot hold. We cannot have $x = f = y$ either, in view of Lemma 3.9. We are thus left with the cases $x \prec f = y$ and $x = f \prec y$, which are also seen to be impossible by putting Lemmas 3.8 and 3.9 together with (†) and (‡).

Finally, we note that essentially the same argument applies to dealing with the cases $f \equiv 0$ and $f \equiv 1$. □

Lemmas 3.7 and 3.10, together with Corollary 3.3, yield:

Corollary 3.11. *X is destructible.*

We shall now see that X satisfies $\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ under the Continuum Hypothesis.

Lemma 3.12. *If $h \in X$ is an accumulation point of a subset $A \subseteq X$, then $h = \sup\{f \in A : f \prec h\}$ or $h = \inf\{f \in A : h \prec f\}$.*

Proof. Let $L = \{f \in A : f \prec h\}$ and $R = \{f \in A : h \prec f\}$. We first deal with the case where L and R are both nonempty.

Let $x = \sup L$ and $y = \inf R$. Clearly, $x \preceq h \preceq y$. If we had $x \prec h \prec y$, it would follow that $(x, y) \cap A \subseteq \{h\}$, thus contradicting the assumption that h is an accumulation point of A. Therefore, $h = x$ or $h = y$.

We can proceed similarly in the cases $L = \emptyset$ and $R = \emptyset$. □

Lemma 3.13. *If a closed subset of $F \subseteq X$ is infinite, then some $h \in F$ is eventually constant.*

Proof. Pick an arbitrary $A \in [F]^{\aleph_0}$. As F is compact, there is a point $h \in F$ that is an accumulation point of A. We can apply Lemma 3.12 and assume, without loss of generality, that $h = \sup\{f \in A : f \prec h\}$. Since $h \notin \{f \in A : f \prec h\}$, it follows from Lemma 3.9 that $\{\xi \in \omega_1 : h(\xi) = 1\}$ is countable. □

Proposition 3.14. CH *implies* $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Let $C = \{f \in {}^{\omega_1}2 : f \text{ is eventually constant}\}$. Using CH, write $C = \{f_\alpha : \alpha \in \omega_1 \setminus \omega\}$.

Now let $(\mathcal{U}_\alpha)_{\alpha\in\omega_1}$ be a sequence of open covers of X. For each $\alpha \in \omega_1 \setminus \omega$, pick $U_\alpha \in \mathcal{U}_\alpha$ such that $f_\alpha \in U_\alpha$. By Lemma 3.13, the set $F = X \setminus \bigcup\{U_\alpha : \alpha \in \omega_1 \setminus \omega\}$ is finite. Thus we can cover all the points in F by choosing one open set U_n in each $\mathcal{U}_n$ with $n \in \omega$, and so we get $(U_\alpha)_{\alpha\in\omega_1}$ such that $U_\alpha \in \mathcal{U}_\alpha$ for all $\alpha \in \omega_1$ and $X = \bigcup\{U_\alpha : \alpha \in \omega_1\}$. □

Note that, in order to conclude $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ in Proposition 3.14, it would suffice to find a subspace $Y \subseteq X$ satisfying $\mathsf{S}_1^{\omega_1}(\mathcal{O}_Y, \mathcal{O}_Y)$ and such that $\mathsf{S}_1^{\omega_1}(\mathcal{O}_{X\setminus U}, \mathcal{O}_{X\setminus U})$ for every open set U with $Y \subseteq U \subseteq X$. The fact that we could get considerably more than this in our example (which is due to Lemma 3.13) suggests that there might still be room for improvement.

Problem 3.15. *Is there such a counterexample in* ZFC*? What about under weaker additional hypotheses,* e.g. MA*?*

Finally, let us make an observation about this space of which we shall make use in the next section:

Lemma 3.16. *If $F \subseteq X$ is closed, then there is $h \in F$ such that $\pi\chi(h, F) = \aleph_0$.*

Proof. If F is finite, the result is immediate. If F is infinite, pick an arbitrary $A \in [F]^{\aleph_0}$; since F is compact, A has an accumulation point $h \in F$. As in the proof of Lemma 3.13, we can apply Lemma 3.12 and assume that $h = \sup\{f \in A : f \prec h\}$. It follows that $\{(f, h) \cap F : f \in A, f \prec h\}$ is a countable π-base for h in F. $\square$

4. Some classes of destructible spaces

Recall that a Hausdorff space is *dyadic* if it is a continuous image of the Cantor cube 2^κ for some infinite cardinal κ.

Theorem 4.1 (Šanin [26])**.** *If a dyadic space has weight λ, then it is a continuous image of 2^λ.*

In what follows we shall make use of the following fact, the straightforward proof of which we omit:

Lemma 4.2. *If a topological space X satisfies $\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ (respectively, One has no winning strategy in $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$), then every closed subspace of X and every continuous image of X also satisfy $\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ (respectively, One has no winning strategy in $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$).*

Lemma 4.3. *The following conditions are equivalent for an infinite cardinal κ:*

(*a*) *2^κ is indestructible;*
(*b*) *$\mathsf{S}_1^{\omega_1}(\mathcal{O}_{2^\kappa}, \mathcal{O}_{2^\kappa})$;*
(*c*) *$\kappa = \omega$.*

Proof. We already know that $(a) \to (b)$ by Theorem 1.4.

For $(b) \to (c)$, suppose to the contrary that κ is uncountable. Then 2^κ includes a copy of 2^{ω_1}, which is closed since 2^{ω_1} is compact. Thus, by Lemma 4.2, $\mathsf{S}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ fails for 2^κ since it fails for 2^{ω_1}: in order to see this, we need only consider for each $\alpha \in \omega_1$ the open cover $\mathcal{U}_\alpha = \{\pi_\alpha^{-1}[\{0\}], \pi_\alpha^{-1}[\{1\}]\}$ of 2^{ω_1}, and no matter how $U_\alpha \in \mathcal{U}_\alpha$ are chosen for $\alpha \in \omega_1$ the set $2^{\omega_1} \setminus \bigcup\{U_\alpha : \alpha \in \omega_1\}$ will consist of exactly one point.

$(c) \to (a)$ is a direct consequence of Theorem 6(a) of [33], which states that every hereditarily Lindelöf space is indestructible. $\square$

We can now characterize the class of dyadic indestructible spaces:

Corollary 4.4. *The following are equivalent for a dyadic space X:*

(*a*) *X is indestructible;*
(*b*) *$\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$;*
(*c*) *X does not include a copy of 2^{ω_1};*
(*d*) *$w(X) = \aleph_0$.*

Proof. Again, $(a) \to (b)$ is clear in view of Theorem 1.4, and $(b) \to (c)$ was already observed in the proof of Lemma 4.3. The implication $(c) \to (d)$ is a particular case of J. Hagler's Theorem 1 of [12]. Finally, Lemmas 4.2 and 4.3 together with Theorem 4.1 yield $(d) \to (a)$. $\square$

This alternative proof of $(a) \to (d)$ in Corollary 4.4 may also be of interest:

Proof of $(a) \to (d)$. It was proven by B. Efimov in [7] that, if a dyadic space X has uncountable weight, then the set of G_δ points of X is not dense in X. Let then Ω be a nonempty open subset of X with $\psi(p, X) > \aleph_0$ for every $p \in \Omega$; we can then use regularity of X to find a nonempty G_δ subset F of Ω such that F is closed in X. Since F is a G_δ in X as well, it follows that $\psi(p, F) = \psi(p, X) > \aleph_0$ for all $p \in F$. By Corollary 3.3, F is destructible; therefore, X is destructible by Lemma 4.2 and Theorem 1.4. $\square$

Corollary 4.5. *Every dyadic space of cardinality greater than* $\mathfrak{c}$ *is destructible.*

Proof. By Lemma 4.1 and Corollary 4.4, we have that an indestructible dyadic space X must be a continuous image of the Cantor space 2^ω, and thus $|X| \leq 2^{\aleph_0} = \mathfrak{c}$. $\square$

Corollary 4.4 tells us that the properties of including a copy of 2^{ω_1} and being destructible are equivalent for dyadic spaces. We shall now try to draw some lines between them for compact Hausdorff spaces in general.

Recall the following theorem of B. E. Šapirovskiĭ [30] (see also 3.18 in [14]):

Theorem 4.6 (Šapirovskiĭ [30]). *The following conditions are equivalent for a compact* T_2 *space* X *and an uncountable cardinal* κ:

- (a) $[0,1]^\kappa$ *is a continuous image of* X;
- (b) 2^κ *is a continuous image of a closed subset of* X;
- (c) *there is a closed nonempty* $F \subseteq X$ *such that* $\pi\chi(x, F) \geq \kappa$ *for all* $x \in F$;
- (d) *there is a* κ*-dyadic system in* X, i.e. *an indexed family* $\langle F^i_\alpha : \alpha \in \kappa, i \in 2\rangle$ *of closed subsets of* X *such that* $F^0_\alpha \cap F^1_\alpha = \emptyset$ *for all* $\alpha \in \kappa$ *and* $\bigcap\{F^{p(\xi)}_\xi : \xi \in \mathrm{dom}(p)\} \neq \emptyset$ *for all* $p \in Fn(\kappa, 2)$.

The case $\kappa = \omega_1$ of Theorem 4.6 is of interest for the study of destructibility of compact spaces:

Lemma 4.7. *Let* X *be a compact space.*

- (a) *If* X *includes a copy of* 2^{ω_1}, *then there is an* ω_1*-dyadic system in* X.
- (b) *If there is an* ω_1*-dyadic system in* X, *then* $\neg\mathsf{S}^{\omega_1}_1(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Since $\langle \pi^{-1}_\alpha[\{i\}] : \alpha \in \omega_1, i \in 2\rangle$ is an ω_1-dyadic system in 2^{ω_1}, it defines an ω_1-dyadic system in X. This proves (a).

For (b), let $\langle F^i_\alpha : \alpha \in \omega_1, i \in 2\rangle$ be an ω_1-dyadic system in X. For each $\alpha \in \omega_1$, let $\mathcal{U}_\alpha = \{X \setminus F^0_\alpha, X \setminus F^1_\alpha\}$. The sequence $(\mathcal{U}_\alpha)_{\alpha\in\omega_1}$ witnesses the failure of $\mathsf{S}^{\omega_1}_1(\mathcal{O}_X, \mathcal{O}_X)$: for every $f \in {}^{\omega_1}2$, we have that $\bigcap\{F^{f(\alpha)}_\alpha : \alpha \in \omega_1\} \neq \emptyset$ since this is the intersection of a centred family of closed subsets of a compact space. $\square$

Let us also evoke Theorem 3.1 of [24] and Corollary 4 of [3]:

Theorem 4.8 (Pierce [24]). *If* X *is an extremally disconnected compact* T_2 *space, then* $w(X)^{\aleph_0} = w(X)$.

Theorem 4.9 (Balcar-Franek [3]). *If* X *is an extremally disconnected compact* T_2 *space, then* X *can be continuously mapped onto the Cantor cube* $2^{w(X)}$.

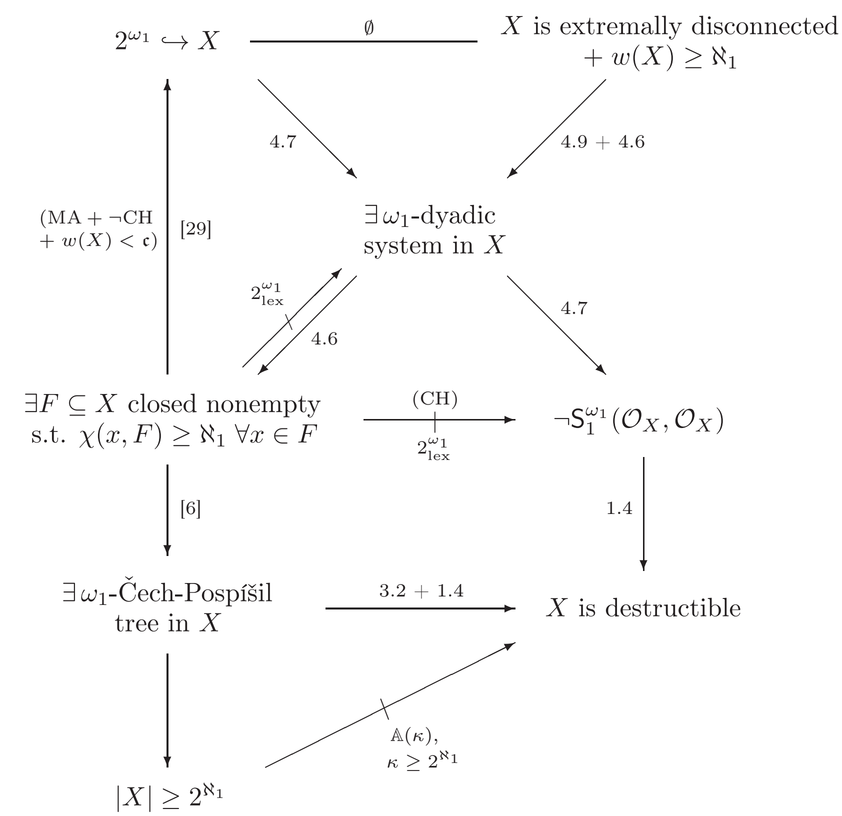
In particular, if X is an extremally disconnected compact T_2 space of uncountable weight, it follows from Theorem 4.8 that $w(X) \geq \mathfrak{c}$; thus, by Theorems 4.9 and 4.6, there is a $\mathfrak{c}$-dyadic system in X — which, in view of Lemma 4.7 and Theorem

1.4, implies that X is destructible.[1] This shows that perfect preimages of compact Rothberger T_2 spaces need not be Rothberger or even indestructible: let X be the one-point compactification of a discrete space of cardinality $\aleph_1$; then the Stone space $\theta(X)$ of the regular closed algebra of X is a perfect preimage of X (see [11, Theorem 3.2]), yet $\theta(X)$ is destructible — by the previous argument — whereas X is Rothberger.

Thus we have the following diagram for X compact T_2:[2]

[1]This fact also follows from Corollary 3.4, since extremally disconnected T_2 spaces have no nontrivial convergent sequences (see [11, Theorem 1.3]).

[2]It is worth remarking that some of the implications do not require compactness of X.



The implication in the diagram that assumes $\mathrm{MA} + \neg\mathrm{CH} + w(X) < \mathfrak{c}$ is a particular case of L. B. Shapiro's Theorem 1.3 of [29]. Each nonimplication is marked with the space that is a counterexample to it. We denote by $\mathbb{A}(\kappa)$ the one-point compactification of the discrete space of size κ, and by $2^{\omega_1}_{\mathrm{lex}}$ the space considered in the proof of Theorem 1.9 presented in Section 3.

The fact that $2^{\omega_1}_{\mathrm{lex}}$ shows the nonimplication that points to "there is an ω_1-dyadic system in X" in the diagram follows from Lemma 3.16 and Theorem 4.6; since this space has weight $\mathfrak{c}$ (see the paragraph preceding Lemma 5.6), we have, in particular, that the weight constraint in Shapiro's theorem is the best possible. This can also be seen from the fact that the properties "$2^{\omega_1} \hookrightarrow X$" and "$X$ is extremally disconnected" cannot hold simultaneously — represented in the diagram by the line labeled with the symbol $\emptyset$ —, which follows from the (already previously mentioned) fact that every convergent sequence in an extremally disconnected Hausdorff space is trivial; thus any extremally disconnected compact Hausdorff space of uncountable weight — *e.g.* $\beta\omega$ — has an ω_1-dyadic system but does not include a copy of 2^{ω_1}. We point out again that Theorem 4.8 implies that any such space would necessarily have weight $\geq \mathfrak{c}$, which is in accordance with Shapiro's result.

From such considerations regarding the space from Section 3 we can derive the following observation concerning Shapiro's theorem:

Proposition 4.10. *Let* (Υ) *denote the statement:*

> *If* X *is compact* T_2 *of weight* $\aleph_1$ *and there is a closed nonempty* $F \subseteq X$ *with* $\chi(x, F) = \aleph_1$ *for all* $x \in F$*, then* X *includes a copy of* 2^{ω_1}.

Then $\mathrm{MA} + \neg\mathrm{CH} \to (\Upsilon) \to \neg\mathrm{CH}$.

Still in the diagram, it is also worth noting that the existence of an ω_1-dyadic system does not follow from the existence of an ω_1-Čech-Pospíšil tree. Note that there is an ω_1-dyadic system in X if and only if there is a sequence $(\mathcal{U}_\alpha)_{\alpha\in\omega_1}$ of two-element open covers of X witnessing the failure of $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$. Similarly, there is an ω_1-Čech-Pospíšil tree in X if and only if One has a winning strategy in $\mathsf{G}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ in which she plays only two-element open covers of X. This raises the following questions:

Problem 4.11. *For* X *compact, is* $\mathsf{S}_1^{\omega_1}(\mathcal{O}_X, \mathcal{O}_X)$ *equivalent to the nonexistence of an* ω_1*-dyadic system in* X*?*

Problem 4.12. *Is destructibility of a space* X *equivalent to the existence of an* ω_1*-Čech-Pospíšil tree in* X*?*

5. Indestructibility and large cardinals

In the paper [35], the following results were established:

Theorem 5.1 (Tall-Usuba [35])**.** *If it is consistent with* ZFC *that there is an inaccessible cardinal, then it is consistent with* ZFC *that every Lindelöf* T_3 *indestructible space of weight* $\leq \aleph_1$ *has size* $\leq \aleph_1$.

Theorem 5.2 (Tall-Usuba [35])**.** *If it is consistent with* ZFC *that there is an inaccessible cardinal, then it is consistent with* ZFC *that the* $\aleph_1$*-Borel Conjecture holds.*

Theorem 5.3 (Tall-Usuba [35])**.** *If it is consistent with* ZFC *that there is a weakly compact cardinal, then it is consistent with* ZFC *that there is no Lindelöf* T_1 *space of pseudocharacter* $\leq \aleph_1$ *and size* $\aleph_2$.

We will show that the large cardinal assumptions are necessary in the above theorems by proving:

Theorem 5.4. *If* KH *holds, then there is a compact* T_2 *indestructible space of weight* $\leq \aleph_1$ *and size* $> \aleph_1$.

Theorem 5.5. *If* ω_2 *is not weakly compact in* $\mathbf{L}$*, then there is a Lindelöf* T_3 *indestructible space of pseudocharacter* $\leq \aleph_1$ *and size* $\aleph_2$.

Both of these results were obtained independently, around the same time as we did, by T. Usuba. We are very thankful to Stevo Todorčević for bringing to our attention that the construction found in [37] would be useful for the topic we discuss in this section.

We point out that indestructibility in Theorem 5.5 is relevant in view of the fact that the existence of an indestructible counterexample to the conclusion of Theorem 5.3 is shown in [35] by adding $\aleph_3$ Cohen subsets of ω_1 under the hypothesis that ω_2 is not weakly compact in $\mathbf{L}$.

The following construction is taken from Section 8 of Todorčević's article [37]. Let T be a rooted Hausdorff tree such that all the levels and branches of T have size $\leq \aleph_1$. We may assume that T is an initial part of $\langle {}^{<\omega_2}(\omega_1+1), \subseteq\rangle$ satisfying the following condition:

$(*)$ for each $t \in T$, the set $\{\xi \leq \omega_1 : t^\frown(\xi) \in T\}$ is a successor ordinal.

Let $\prec$ be the ordering on the set $L_T = \{\bigcup B : B \text{ is a branch of } T\}$ naturally induced by the lexicographical ordering of the branches of T, and regard L_T as a linearly ordered topological space. The argument present in Lemma 8.1(i) of [37] applies in this case to show that $w(L_T) \leq |T|$. For each $t \in T$, let $L_T(t) = \{s \in L_T : t \subseteq s\}$.

Lemma 5.6 (see [37])**.** *$L_T(t)$ is a compact subspace of L_T for all $t \in T$.*

Proof. We will prove that $L_T = L_T(\emptyset)$ is compact; the general case is analogous. Since L_T is a linearly ordered space, compactness of L_T is equivalent to completeness of the linear ordering of L_T.

Let $A \subseteq L_T$ be arbitrary. For each $s \in L_T$, let $\tilde{s} = s \cup \{\langle \xi, 0\rangle : \xi \in \omega_2 \setminus \mathrm{dom}(s)\} \in {}^{\omega_2}(\omega_1+1)$. Now define $f \in {}^{\omega_2}(\omega_1+1)$ recursively by

$$f(\alpha) = \sup\{\tilde{s}(\alpha) : s \in A \text{ and } s \restriction \alpha = f \restriction \alpha\}$$

for all $\alpha \in \omega_2$. There must be some $\beta \in \omega_2$ with $f \restriction \beta \notin T$, since otherwise $\{f \restriction \beta : \beta \in \omega_2\}$ would be a branch of cardinality $\aleph_2$ in T. Let then β_0 be the least such β, and define $u = f \restriction \beta_0$.

Case 1. β_0 is a limit ordinal. It follows from the minimality of β_0 that $\{u \restriction \xi : \xi \in \beta_0\} \subseteq T$; therefore, as $u \notin T$ and β_0 is a limit ordinal, we have that $\{u \restriction \xi : \xi \in \beta_0\}$ is a branch of T. Hence $u = \bigcup\{u \restriction \xi : \xi \in \beta_0\} \in L_T$. Note that $f = \tilde{u}$, by the construction of f. It follows that $\sup A = u \in L_T$.

Case 2. β_0 is a successor ordinal. Let $\alpha \in \omega_2$ be such that $\beta_0 = \alpha+1$, and define $v = u \restriction \alpha \in T$. Since $u \notin T$ and T satisfies $(*)$, it follows from the construction of f that $u(\alpha) = 0$. Thus $\{v \restriction \xi : \xi \leq \alpha\}$ is a branch of T (note that, by $(*)$, if $v^\frown(0) \notin T$ then v has no extension in T), whence $v = \bigcup\{v \restriction \xi : \xi \leq \alpha\} \in L_T$. As in the previous case, $f = \tilde{v}$ and therefore $\sup A = v \in L_T$. □

Lemma 5.7. $\psi(L_T) \leq \aleph_1$.

Proof. Fix an arbitrary $s \in L_T$. It suffices to show that there is $A \subseteq \{s' \in L_T : s' \prec s\}$ such that $|A| \leq \aleph_1$ and $(a, s) = \emptyset$, where $a = \sup A \in L_T$; a similar argument will show that the analogous condition holds also to the right of s.

Let $S = \{\xi \in \mathrm{dom}(s) : \exists s' \in L_T\ (s' \prec s \text{ and } s' \restriction \xi = s \restriction \xi)\}$.

Case 1. S does not have a greatest element. For each $\xi \in S$, pick $r_\xi \in L_T$ with $r_\xi \prec s$ and $r_\xi \restriction \xi = s \restriction \xi$. It is clear that, if $s' \in L_T$ is such that $s' \prec s$, then $s' \prec r_\xi$ for some $\xi \in S$. Thus we can take $A = \{r_\xi : \xi \in S\}$.

Case 2. There is $\beta = \max S$. Since $\{s \restriction \alpha : \alpha \in \mathrm{dom}(s)\}$ is a branch of T, we must have that $\beta + 1 \in \mathrm{dom}(s)$; let then $\theta = s(\beta+1)$. For each $\eta \in \theta$, let $r_\eta = \sup\{s' \in L_T : s' \restriction \beta = s \restriction \beta \text{ and } s'(\beta+1) = \eta\} \prec s$. Then $A = \{r_\eta : \eta \in \theta\}$ is as required. □

The next result will be the main tool for obtaining indestructibility of our examples.

Lemma 5.8. *The following conditions are equivalent:*

(*a*) *L_T is destructible;*

(*b*) *there is a countably closed forcing that adds a new Dedekind cut in* $\langle L_T, \prec\rangle$*;*
(*c*) *there is a countably closed forcing that adds a new branch in* T*;*
(*d*) T *has a subtree isomorphic to* $\langle {}^{<\omega_1}2, \subseteq\rangle$*;*
(*e*) *there is an* ω_1*-Čech-Pospíšil tree in* L_T.

Proof. Since L_T is a linearly ordered compact space, it fails to be compact in a generic extension by a countably closed partial order if and only if this partial order adds in L_T a new Dedekind cut with no least upper bound. Thus the equivalence between (*a*) and (*b*) follows from the observation that every Dedekind cut in the extension that has a least upper bound was already in the ground model.

For (*b*) $\to$ (*c*), let $A \subseteq L_T$ be a Dedekind cut in a countably closed forcing extension, and suppose $A \notin \mathbf{M}$. Let α be the least ordinal $\leq \omega_2$ such that there is no $t_\alpha \in T$ satisfying $\mathrm{dom}(t_\alpha) = \alpha$, $\{a \in A : t_\alpha \subseteq a\} \neq \emptyset$ and $\{b \in L_T \setminus A : t_\alpha \subseteq b\} \neq \emptyset$. Let also $C = \{t_\beta : \beta \in \alpha\}$ and $u = \bigcup C$. We claim that $u \notin \mathbf{M}$, which implies $C \notin \mathbf{M}$.

Suppose, to the contrary, that $u \in \mathbf{M}$.

Case 1. α is a successor ordinal. Then there is $\eta \in \omega_1 + 1$ such that $\{\xi \in \omega_1 : \exists a \in A\ (u^\frown(\xi) \subseteq a)\} = \eta$ and $u^\frown(\eta) \subseteq b$ for some $b \in L_T \setminus A$, hence $A = \{s \in L_T : s \prec u^\frown(\eta)\} \in \mathbf{M}$, a contradiction.

Case 2. α is a limit ordinal. Let $E = \{s \in L_T : u \subseteq s\}$. Then either $E \subseteq A$ or $E \subseteq L_T \setminus A$, for otherwise we could have defined $t_\alpha = u$. Therefore we have either $A = \{s \in L_T : \exists s' \in L_T\ (u \subseteq s' \text{ and } s \preceq s')\}$ or $A = \{s \in L_T : s \prec u\}$, thus contradicting the assumption that $A \notin \mathbf{M}$.

Hence $u \notin \mathbf{M}$; in particular, this implies $\mathrm{cf}(\alpha) > \omega$, since the forcing is countably closed. Note that $C \notin \mathbf{M}$ must be a branch of T, for otherwise we would have $u \in T$ and therefore $u \in \mathbf{M}$.

The implication (*c*) $\to$ (*d*) follows from essentially the same argument present in the proof of Lemma 4.3 of [36].

For (*d*) $\to$ (*e*), let $\{t_p : p \in {}^{<\omega_1}2\} \subseteq T$ be such that $p \subseteq q \leftrightarrow t_p \subseteq t_q$ for all $p, q \in {}^{<\omega_1}2$. For each $p \in {}^{<\omega_1}2$, define $F_p = L_T(t_p)$, which is closed by Lemma 5.6. Now, for each $h \in {}^{\omega_1}2$, let $F_h = \bigcap\{F_{h\restriction\alpha} : \alpha \in \omega_1\}$; note that $F_h \neq \emptyset$ since L_T is compact. Then $\langle F_r : r \in {}^{\leq\omega_1}2\rangle$ is an ω_1-Čech-Pospíšil tree in L_T.

Finally, (*e*) $\to$ (*a*) follows from Proposition 3.2 and Theorem 1.4. $\square$

Note that Lemma 5.8 extends Corollary 3.11.

We now make the following observation, which is essentially taken from Silver's Lemma ([32]; see also *e.g.* [18, Lemma VIII.3.4]):

Lemma 5.9. *If* T *is an* ω_1*-tree, then* T *does not have a subtree isomorphic to* $\langle {}^{<\omega_1}2, \subseteq\rangle$.

Proof. Let $\langle T, \leq\rangle$ be an ω_1-tree, and suppose, to the contrary, that $\{t_p : p \in {}^{<\omega_1}2\} \subseteq T$ is such that $p \subseteq q \leftrightarrow t_p \leq t_q$ for all $p, q \in {}^{<\omega_1}2$. Let $\theta = \sup\{\mathrm{ht}_T(t_p) : p \in {}^{<\omega}2\} \in \omega_1$ and, for each $f \in {}^{\omega}2$, pick $u_f \in T_\theta$ such that t_f and u_f are $\leq$-comparable (this is possible since there is $p \in {}^{<\omega_1}2$ extending f with $\mathrm{ht}_T(t_p) > \theta$). The mapping $f \mapsto u_f$ is injective, which contradicts the fact that T_θ is countable. $\square$

Now we can prove Theorem 5.4:

Proof of Theorem 5.4. Let T be a Kurepa tree with $\kappa > \aleph_1$ cofinal branches. We may assume that T is Hausdorff and rooted. By Lemmas 5.9 and 5.8, the linearly

ordered compact space L_T is indestructible. Furthermore, $w(L_T) \leq |T| = \aleph_1$ and $|L_T| \geq \kappa > \aleph_1$. $\square$

Here we remark that, although the space L_T obtained from a Kurepa tree is indestructible, the tree itself can of course fail to remain Kurepa in a countably closed forcing extension, *e.g.* if the forcing collapses $2^{\aleph_1}$ onto $\aleph_1$.

Corollary 5.10. *The existence of an inaccessible cardinal and the statement "every Lindelöf T_3 indestructible space of weight $\leq \aleph_1$ has size $\leq \aleph_1$" are equiconsistent.*

Proof. From Theorems 5.1 and 5.4. $\square$

Corollary 5.11. *The $\aleph_1$-Borel Conjecture and the existence of an inaccessible cardinal are equiconsistent.*

Proof. Since Tychonoff spaces of weight $\leq \aleph_1$ are embeddable in $[0,1]^{\omega_1}$, the result follows from Theorems 5.2 and 5.4. $\square$

We also have:

Corollary 5.12. *For a regular cardinal $\kappa > \aleph_1$, it is consistent with* ZFC *that there is a compact T_2 indestructible space of weight $\leq \aleph_1$ and size $2^{\aleph_1} = \kappa$.*

Proof. Since it is consistent that $2^{\aleph_1} = \kappa$ + "there is a Kurepa tree with κ cofinal branches", we can proceed as in the proof of Theorem 5.4. $\square$

Now we turn to Theorem 5.5. The main ingredient needed in the proof we shall present for it is the following combination of Theorem 6.1 of [13] (see 1.10 in [38]) and Theorem 3.9 of [17]:

Theorem 5.13 (Jensen [13], König [17])**.** *If a regular uncountable cardinal κ is not weakly compact in $\mathbf{L}$, then there is a sequence $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\kappa)}$ of functions $f_\alpha : \alpha \to \alpha$ that is* coherent — i.e. *satisfies $f_\alpha =^* f_\beta \restriction \alpha$ for every $\alpha, \beta \in \lim(\kappa)$ with $\alpha < \beta$ — and such that $\langle T(\mathcal{F}), \subseteq \rangle$ is a κ-Aronszajn tree, where $T(\mathcal{F}) = \bigcup_{\xi \in \kappa} \bigcup_{\alpha \in \lim(\kappa) \setminus \xi} \{f \in {}^{\xi}\xi : f =^* f_\alpha \restriction \xi\}$.*

The following lemma will be essential in what we shall do next.

Lemma 5.14. *If $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ is coherent, then no branch of $\langle T(\mathcal{F}), \subseteq \rangle$ has cofinality ω_1.*

Proof. Suppose, to the contrary, that $B \subseteq T(\mathcal{F})$ is a branch of cofinality ω_1. Let $g = \bigcup B$ and $\gamma = \mathrm{dom}(g) \in \lim(\omega_2)$, and fix a strictly increasing sequence of limit ordinals $(\gamma_\eta)_{\eta \in \omega_1}$ with $\sup\{\gamma_\eta : \eta \in \omega_1\} = \gamma$. Note that for each $\eta \in \omega_1$ we have $g \restriction \gamma_\eta \in T(\mathcal{F})$, and thus $g \restriction \gamma_\eta =^* f_{\gamma_\eta} =^* f_\gamma \restriction \gamma_\eta$ since $\mathcal{F}$ is coherent. This yields

$$\omega_1 = \bigcup_{k \in \omega} \{\eta \in \omega_1 : |\{\xi \in \gamma_\eta : g(\xi) \neq f_\gamma(\xi)\}| = k\};$$

thus, by regularity of ω_1, there is $k_0 \in \omega$ such that $|\{\xi \in \gamma_\eta : g(\xi) \neq f_\gamma(\xi)\}| = k_0$ for uncountably many $\eta \in \omega_1$. Since the mapping $\eta \mapsto |\{\xi \in \gamma_\eta : g(\xi) \neq f_\gamma(\xi)\}|$ is nondecreasing, we must have $|\{\xi \in \gamma_\eta : g(\xi) \neq f_\gamma(\xi)\}| = k_0$ for every $\eta \in \omega_1$ greater than some $\eta_0 \in \omega_1$. But then $|\{\xi \in \gamma : g(\xi) \neq f_\gamma(\xi)\}| = k_0$, which implies $g \in T(\mathcal{F})$, thus contradicting the choice of B. $\square$

The next lemma will guarantee indestructibility of the space we shall obtain.

Lemma 5.15. *Assume* CH. *If* $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ *is coherent, then* $\langle T(\mathcal{F}), \subseteq \rangle$ *does not have a subtree isomorphic to* $\langle {}^{<\omega_1}2, \subseteq \rangle$.

Proof. This is pretty much like Lemma 5.9. Suppose that there is $\{g_s : s \in {}^{<\omega_1}2\} \subseteq T(\mathcal{F})$ such that $g_s \subseteq g_t \leftrightarrow s \subseteq t$ for all $s, t \in {}^{<\omega_1}2$. By CH, we have that $\delta = \sup\{\mathrm{dom}(g_s) : s \in {}^{<\omega_1}2\} \in \omega_2$. For each $h \in {}^{\omega_1}2$, let $g_h = \bigcup\{g_{h \restriction \alpha} : \alpha \in \omega_1\}$, and then define $\tilde{g}_h = g_h \cup (f_\delta \restriction (\delta \setminus \mathrm{dom}(g_h)))$; note that $g_h \in T(\mathcal{F})$ by Lemma 5.14, and thus $\tilde{g}_h \in T(\mathcal{F})$. But the δ-th level of $T(\mathcal{F})$ is the set $\{f \in {}^\delta\delta : f =^* f_\delta\}$, which has cardinality $\aleph_1$; this leads to a contradiction, since this set must include $\{\tilde{g}_h : h \in {}^{\omega_1}2\}$ and $h \mapsto \tilde{g}_h$ is one-to-one. □

The following proposition will then be the core of our proof of Theorem 5.5:

Proposition 5.16. *If* CH *holds and* ω_2 *is not weakly compact in* $\mathbf{L}$, *then there is a compact* T_2 *indestructible space of pseudocharacter* $\aleph_1$ *and size* $\aleph_2$.

Proof. Assume that ω_2 is not weakly compact in $\mathbf{L}$, and let then $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ be given by Theorem 5.13. Now consider the linearly ordered compact space L_T, where T is a tree isomorphic to $T(\mathcal{F})$ that satisfies the assumptions stated in the paragraph preceding Lemma 5.6. By Lemmas 5.15 and 5.8, L_T is indestructible. On the one hand, the fact that T is ω_2-Aronszajn implies that $|L_T| \geq \aleph_2$; on the other hand, in view of Lemma 5.14, it also implies that every branch of T has countable cofinality, and thus $L_T \subseteq \{\bigcup C : C \in [T]^{\leq \aleph_0}\}$; since $|T| = \aleph_2$, this yields $|L_T| \leq \aleph_2{}^{\aleph_0} = \aleph_1{}^{\aleph_0} \cdot \aleph_2 = \aleph_2$ by CH. Therefore, $|L_T| = \aleph_2$. Finally, $\psi(L_T) \leq \aleph_1$ by Lemma 5.7, and so $\psi(L_T) = \aleph_1$ since otherwise we would contradict Arhangelskiĭ's Theorem [1]. □

Now Theorem 5.5 follows directly from Proposition 5.16:

Proof of Theorem 5.5. If CH fails, then any subspace $X \subseteq \mathbb{R}$ with $|X| = \aleph_2$ will satisfy the required conditions (recall that, by Theorem 6(a) of [33], every hereditarily Lindelöf space is indestructible). If CH holds, the result follows from Proposition 5.16. □

Corollary 5.17. *The existence of a weakly compact cardinal and the statement "there is no Lindelöf* T_1 *space of pseudocharacter* $\leq \aleph_1$ *and size* $\aleph_2$*" are equiconsistent.*

Proof. From Theorems 5.3 and 5.5. □

We conclude with the following observation:

Proposition 5.18. *Assume that* κ *is an inaccessible cardinal that is not weakly compact in* $\mathbf{L}$. *If* κ *is Lévy-collapsed to* $\aleph_2$, *the model* $\mathbf{M}[G]$ *thus obtained satisfies:*

(i) *every Lindelöf* T_3 *indestructible space of weight* $\leq \aleph_1$ *has size* $\leq \aleph_1$*; and*
(ii) *there is a compact* T_2 *indestructible space with pseudocharacter* $\aleph_1$ *and size* $\aleph_2$.

Proof. Theorem 5.1 is proven in [35] by showing that (i) holds in the model obtained from the Lévy collapse of an inaccessible to $\aleph_2$. Thus we only need to check (ii).

We shall do so by making use of Proposition 5.16. Since CH holds in $\mathbf{M}[G]$, it suffices to show that, in $\mathbf{M}[G]$, ω_2 is not weakly compact in $\mathbf{L}$.

We work within $\mathbf{M}$. As κ is not weakly compact in $\mathbf{L}$, there is in $\mathbf{L}$ a κ-Aronszajn tree $\langle T, \prec \rangle$.

Claim. The Lévy-collapsing partial order $\mathbb{P}$ does not add any cofinal branch in T.

We will argue as in Theorem 9 of [20]. Suppose, in order to obtain a contradiction, that there is a $\mathbb{P}$-name σ with

$$\Vdash \sigma : \check{\kappa} \to \check{T} \text{ is such that } \forall \alpha, \beta \in \check{\kappa}\ (\alpha < \beta \to \sigma(\alpha) \check{\prec} \sigma(\beta)).$$

For each $\alpha \in \kappa$, fix $p_\alpha \in \mathbb{P}$ and $t_\alpha \in T$ such that $p_\alpha \Vdash \sigma(\check{\alpha}) = \check{t}_\alpha$. By the same Δ-system argument used in Lemma VII.6.10 of [18], it follows that there is a subset $A \subseteq \kappa$ with $|A| = \kappa$ such that $\{p_\alpha : \alpha \in A\}$ is pairwise compatible. For arbitrary $\alpha, \beta \in A$ with $\alpha < \beta$, if $q \in \mathbb{P}$ is a common extension of p_α and p_β, we have $q \Vdash \check{t}_\alpha \check{\prec} \check{t}_\beta$; hence, by absoluteness, $t_\alpha \prec t_\beta$. Therefore $\{t_\alpha : \alpha \in A\}$ is a chain of length κ in T, which contradicts the fact that T is κ-Aronszajn. This proves the claim.

We now move to the extension $\mathbf{M}[G]$. By the previous claim, T is an ω_2-Aronszajn tree, and hence it is ω_2-Aronszajn in $\mathbf{L}$. Therefore, ω_2 is not weakly compact in $\mathbf{L}$. □

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